1. Analytic brancches of eigenvalues amd eigenforms

(Summary)

Suppose M^n closed smooth manifold, $f : M \to \mathbb{R}$ smooth map , g a Riemannian metric. $\Omega^q(M)$ is a Frechet space with the family of norms $||\cdot||_r$ provided by maximizing the first r derivatives or by the norms provided by the scalar products $(\omega, \omega')_{2r}$.

For any t the Witten Laplacian $\Delta^f_q(t) : \Omega^q(M) \to \Omega^q(M)$ is a second order formally selfadjoint operator given by

$$\Delta^f_q(t) = \Delta_q + t\mathcal{L}_q + t^2||grad\ f||^2$$

where

$$\mathcal{L}_q = L_X + L^\sharp_X,$$

$X = gradf$. Recall $\mathcal{L}_q$ is order zero operator .

Suppose $M = \mathbb{R}^n$, the metric g and the Morse function f have polynomial growth to ∞ and $\Omega^q(\mathbb{R}^n)_S$, is the subspace of $\Omega^q(\mathbb{R}^n)$ consisting of $q-$forms whose coefficients are functions in S (i.e. rapidly decaying to infty). The scalar product (ω, ω') is well defined and $\Delta_q(f)$ leavs the space $\Omega^q(\mathbb{R}^n)_S$ invariant. Restricted to this space it is formally self adjoint and non negative. It can be regarded as an unbounded operator on the Hilbert space completion of $\Omega^q(\mathbb{R}^n)_S$.

In both cases $\Omega^q(M^n)$, for M^n closed manifold, or $\Omega^q(R^n)_S$ one has

Theorem 0.1 *(Rellich-Kato) For any q there exists the collection of pairs of analytic maps in $t \in \mathbb{R}$ $(\lambda^q_\alpha(t) \in \mathbb{R}, \omega^q_\alpha(t) \in \Omega^q(M))$, with α in a countable collection of indices $\alpha \in \mathcal{A}_q$ called analytic branches, each having holomorphic extension to an open neighborhood of $\mathbb{R}$ inside $\mathbb{C}$,such that :*

1. *$\Delta_q(t)(\omega_\alpha(t)) = \lambda^q_\alpha(t)\omega_\alpha(t)$, $||\omega_\alpha(t)|| = 1$ with $\lambda_\alpha(t)$ exhausting the set of all eigenvalues of $\Delta_q(t)$ with their multiplicity and $\omega^q_\alpha(t)$ forming a complete set of orthonormal eigenvectors for the Hilbert space completion of $\Omega^q(M)$, resp. of $\Omega^q(R^n)_S$.*

2. *$\lambda^q_\alpha(t_0) = 0$ for one t_0 implies $\lambda^q_\alpha(t) = 0$ for all t and exactly $dimH^q(M : \mathbb{R})$ resp. $\dim H^q(S^n; \mathbb{R})$[1] eigenbranches $\lambda^q_\alpha(t)$ which are identically zero and all other are strictly positive for any t.*

3. *It can be shown, see reference below (Haller), that $\lambda^q_\alpha(t) \leq O(t^2)$ and $\lim_{t\to\infty} \frac{\lambda^q_\alpha}{t^2} = \mu^q_\alpha \in \mathbb{R}$.*

2. Morse function case

Suppose that M^n is closed, $f : M \to \mathbb{R}$ is a Morse function and in the neighborhood of any critical point there exists a Morse chart with the Riemannian metric g given by $g_{i,j} = \delta_{i,j}$. Under these hypotheses the following holds.

Theorem 0.2

1. *$\mu^q_\alpha = 0$ and either $\lim_{t\to\infty} \frac{\lambda^q(t)}{t} = \infty$ (cojecturally never) or $\lim_{t\to\infty} \frac{\lambda^q(t)}{t} = 2N$, $N = 0, 1, 2, \cdots$.*

2. *The set $\{\alpha \mid \lim_{t\to\infty} \frac{\lambda^q(t)}{t} = 2N\}$ is finite of specified cardinality $\mathcal{N}(n, q, N; c_0, c_1, \cdots c_n)$ with $c_k = \sharp Cr_k(f)$ as given in reference no 4.*

3. *For each α with $\lim_{t\to\infty} \frac{\lambda^q(t)}{t} = 2N$, $\omega^q_\alpha(t)$ concentrates in the neighborhood of the set of critical points of index q.*

[1] S^n is viewed as the one point compactification of R^n

To derive the above one uses *Gap in the spectrum* lemma below and explicit calculations of the eigenvalues branches of eigenvalues and eigenforms for the model operators described below.

Lemma 0.3 *(Gap in the spectrum) Suppose $A : H \rightsquigarrow H$ is a possibly unbounded self adjoint nonnegative operator in the Hilbert space and $a < b$ $a, b \in \mathbb{R}$. Suppose that H_1, H_2 are two closed subspaces s.t. $H = H_1 \oplus H_2$ and*

1. $H_1 \subset \text{Domain of A}$ *with* $(Av_1, v_1) \leq a||v_1||^2$ *for* $v_1 \in H_1$
2. $(Av_2, v_2) \geq b||v_2||^2$ *for* $v_2 \in H_2 \cap \text{Domain of A}$

Then $SpectA \cap (a, b) = \emptyset$

Moreover if $\dim H_1$ *is finite then* $\sharp\{SpectA \cap [0, a]\} = \dim H_1$.

We apply this to $\Delta_q(t)$ or to $(1/t)\Delta_q(t)$ viewed as an unbounded selfadjoint operator on the L_2-completion of $\Omega^q(M)$ whose domain is $\Omega^q(M)$ with H_1 defined by a finite collection of smooth $q-$forms and H_2 its orthogonal complement,

3.**Model operator.**

Consider $M^n = \mathbb{R}^n$ equipped with the metric $g_{i,j} = \delta_{i,j}$ and the smooth function $f_k := c - 1/2 \sum_{1 \leq i \leq k} x_i^2 + 1/2 \sum_{k+1 \leq i \leq n} x_i^2$ in which case

$$\Delta_q^{n,k}(t) := \Delta_q^{f_k}(t) = -\sum_{i=1}^{n} \partial^2 / \partial x_i^2 + t\mathcal{L}_q^{n,k} + t^2 \sum_{i=1}^{n} x_i^2$$

with $\mathcal{L}_q^{n,k} := L_{gradf_k} + L^\sharp_{gradf_k}$. Standard calculations implies that $\mathcal{L}_q^{n,k}$ is a zero-order operator with constant coefficients, hence given by a $(n, q) \times (n, q)$ square matrix with (n, q) the binomial coefficient, $(n, k) = \dim \wedge^k(\mathbb{R}^n)$.

As a second order operator on $\Omega^q(\mathbb{R}^n)_S$ $(1/t)\Delta_q^k(t)$ is formally self adjoint with the following properties.

1. The eigenvalues of $\lambda_\alpha^q(t)$ of $(1/t)\Delta_q^k(t)$ are constant in t, i.e. = $\lambda_\alpha^q(t) = \lambda_\alpha^q \in \text{even nonnegative interes}$
2. For any even integer $2N$ the set $\{\alpha \mid \lambda_\alpha^q = 2N\}$ is finite of cardinality given by a precise formula (see reference 4)
3. The eigenforms $\omega_\alpha^q(t)$ is a product of the type

$$\prod_{i=1,\cdots n} e^{-tx_i^2} H_r(\sqrt{t}x_i)$$

where $H_r(y)$ is the $r-$th Hermite polynomial, hence concentrated near the critical point of f_k which is zero.

Denote by D_l the disc of radius l in $\mathbb{R}^n$ centered at $0 \in R^n$ and for $l < l'$ denote by $\chi_{l,l'}(t)$ the cutoff smooth function s.t. $\chi_{l,l'}(t) = \begin{cases} 1, \text{if t} \leq \text{l} \\ 0, \text{if t} > \text{l}' \end{cases}$ and by $\tilde{\omega}_\alpha^q(x) := \chi(|x|)\omega_\alpha^q(x)$.

Explicit calculations show that the equality

$$(\Delta_q^k(t)\omega_\alpha^q(t), \omega_\alpha^q(t)) = \lambda_\alpha^q ||\omega_\alpha^q(t)||^2,$$

implies that for $0 < l < l'$ and $\epsilon \in (0,1)$ there exists T depending on l, l' and ϵ such that for $t > T$ one has

$$(\Delta_q^k(t)\tilde{\omega}_\alpha^q(t), \tilde{\omega}_\alpha^q(t)) \leq (\lambda_\alpha^q + e^{-t^\epsilon})||\tilde{\omega}_\alpha^q(t)||^2$$

and

$$(\Delta_q^k(t)\tilde{\omega}_\alpha^q(t), \tilde{\omega}_\alpha^q(t)) \geq (\lambda_\alpha^q - t^{-\epsilon})||\tilde{\omega}_\alpha^q(t)||^2.$$

For a Morse function $f : M \to \mathbb{R}$ one reffers to $\oplus_{x \in Cr(f)} \Delta_q^{k(x)}(t)$, $k(x) = index x$, on the Frechet space $\oplus_{x \in Cr(f)} \Omega^q(\mathbb{R}^n)_S$ as the **model operator of the Morse function** f .

Ideea of the proof of Theorem 0.2

consists in comparing the spectral package of $\Delta_q(t)$ (= branches of eigenvalues and eigenforms) to the spectral package of the model operator of the Morse function f.